Exercise Sheet 4

Question 1.

(a) Let $G = \langle x \rangle$ be a finite cyclic p-group of order p^n. Calculate $\mathrm{Aut}(G)$ in the following steps:

  (i) Show that $G = \langle x^m \rangle$ if and only if $(m, p^n) = 1$. Conclude that $|\mathrm{Aut}(G)| = \phi(n) = p^{n-1}(p-1)$, where ϕ is *Euler's totient function* which enumerates all natural numbers which are smaller than p^n and coprime to n.

  (ii) Prove that $\mathrm{Aut}(C_p) \cong C_{p-1}$, and that $\mathrm{Aut}(C_4) = \langle a \rangle$ where $a : x \to x^{-1}$.

  (iii) Show that if $p = 2$ and $n \geqslant 3$ then $\mathrm{Aut}(C_{2^n}) \cong C_{2^{n-2}} \times C_2$, generated by the maps $a_1 : x \to x^5$ and $a_2 : x \to x^{-1}$.

  (iv) Show that if p is odd then $\mathrm{Aut}(C_{p^n})$ is cyclic.

(b) Let G be a finite p-group with $P \leq G$ such that P is extraspecial. Show that if $[G, P] \leq Z(P)$ then $G = PC_G(P)$.

(c) Let G be a finite *homocyclic* p-group. That is, G is a direct product of n isomorphic cyclic p-groups for some $n \in \mathbb{N}$. Show that $\mathrm{Aut}(G) = NH$ where N is normal p-subgroup of $\mathrm{Aut}(G)$ and $H \cong \mathrm{GL}_n(p)$.

Question 2.

(a) Let G be a finite group and A a group acting faithfully on G. Show that if there is an A-invariant chain of subgroups $G_0 = \Phi(G) \trianglelefteq \ldots G_n = G$ and $B \leq A$ such that $[G_i, B] \leq G_{i-1}$ for $i \in \{1, \ldots, n\}$, then $B \leq O_p(A)$.

(b) Let G be a finite p-group, A a group acting on G and $N \trianglelefteq G$ an A-invariant subgroup. Show that if $a \in A$ is such that $[G, a] \leq N$ and $[N, a] = \{e\}$ then $a^p = e$.

(c) Let G be a finite p-group and A a a group acting faithfully on G. Suppose that there is $a \in A$ such that $a^3 = e$ and a acts fixed point freely on G. Show that G is nilpotent of class at most two in the following steps:

  (i) Prove that $(g \cdot a^2)(g \cdot a)g = e$ for all $g \in G$ and use this to deduce that $[g, g \cdot a] = e$ for all $g \in G$.

  (ii) Show that $(|G|, 3) = 1$.

  (iii) Let $c_x : G \to G$ such that $g \cdot c_x = g^x$. Show that $c_x a$ is fixed point free on G and conclude that $[g^x, g \cdot a] = [g^x, g \cdot a^2] = e$.

  (iv) From (iii), deduce that g^x commutes with g so that $[x, g, g] = e$ for all $x, g \in G$. (So G is a 2-*Engel* group).

  (v) Prove that for all $g, h, k \in G$ we have that $[g, hk] = [g, h][g, k][g, h, k]$ and $[g, h^{-1}] = [g, h]^{-1} = [h, g]$. Conclude that $[g, h, k] = [k, g, h]$.

(vi) Apply the Hall–Witt identity (Proposition 0.41 (iii) in the prerequisites) to deduce that $[g, k, h]^3 = e$ for all $g, h, k \in G$, and conclude that G has class two.

Question 3.
Let G be a finite p-group, for p an odd prime, and A a finite group acting faithfully on G with $(|A|, p) = 1$.

(a) Show that if p is odd then A acts faithfully on $\Omega_1(G)$.

(b) Show that A acts faithfully on $J_a(G)$. Does A act faithfully on $J_e(G)$?

(c) Show that the action of A commutes with the p^i-power map $\phi_i : G \to G$ where $g \mapsto g^{p^i}$. (Conclude that $G/\Phi(G)$ and $\Omega_1(G)$ are isomorphic as $\mathbb{F}_p A$-modules.)

Question 4.
Let G be a finite p-group of nilpotency class two, where p is odd, and let A be a finite group acting faithfully on G with $(|A|, p) = 1$.

(a) Show that for all $g \in G$, there is a unique x such that $x^2 = g$ (we shall write $x = \sqrt{g}$).

(b) Show that the following hold:

(i) if $g, h \in Z(G)$ then $\sqrt{g}, \sqrt{h} \in Z(G)$;

(ii) $g\sqrt{g^{-1}} = \sqrt{g}$ for $g \in G$; and

(iii) $\sqrt{g \cdot a} = \sqrt{g} \cdot a$ for all $a \in A$.

(c) Define the binary operation $+ : G \times G \times G$ such that $g + h = gh\sqrt{[h, g]}$. Show that $(G, +)$ is abelian finite group which inherits an action of A.

(d) Investigate what properties $(G, *)$ and $(G, +)$ share (e.g. exponent, A-invariant subgroups,...).