Exercise Sheet 3

Question 1.

(a) Let G be a finite p-group. Show that if $Z_2(G)$ is cyclic, then either G is cyclic or $p = 2$ and G has an cyclic subgroup of index 2. Using the lecture notes, determine the finite p-groups G for which $Z_2(G)$ is cyclic.

(b) Suppose that G is a finite p-group such that $|G'/\Phi(G')| \leqslant p^2$. Then $[G', G']$ is cyclic and G' has nilpotency class at most 2. (You may use that if $K \cong C_p \times C_p$ then $\mathrm{Aut}(K) \cong \mathrm{GL}_2(p)$ and $|\mathrm{GL}_2(p)|$ is divisible by p, but not by p^2.)

(c) Suppose that G is a finite p-group with $|G| = p^n$ with $n > 1$. Assume that there is $1 < m < n$ such that every subgroup of G of order p^m is cyclic. Prove that either G is either cyclic or generalized quaternion.

Question 2.

(a) Show that $\mathfrak{X}(G)$ is the unique largest subgroup of G admitting a Q-series.

(b) Let G be a finite p-group. Set $\mathcal{D}(G)$ to be the set of abelian subgroups A of G such that if $x \in G$ and $\langle A, x\rangle$ is nilpotency class two, then $[A, x] = \{e\}$ for all $x \in G$. Define $D^*(G) := \langle \mathcal{D}(G)\rangle$.

   (i) Show that $D^*(G)$ is a characteristic subgroup of G.

   (ii) Show that if $D^*(G) \leq H \leq G$ then $D^*(G) \leq D^*(H)$.

   (iii) Show that $D^*(G)$ is abelian.

   (iv) Show that $D^*(G)$ is the unique maximal member of $\mathcal{D}(G)$.

   (v) (Harder) Show that $Z(G) \leq D^*(G) \leq Z(J(G))$ when $p = 2$.

Question 3.

Let G be a p-group and let $A \in \mathcal{A}_e(G)$ have order p^m for some $m \in \mathbb{N}$. For $i \in \{0, \ldots, m-1\}$ set $\mathcal{A}_e^i(G)$ to be the set of abelian subgroups of G of order p^{m-i} and $J_e^i(G) := \langle \mathcal{A}_e^i(G)\rangle$. Then $\mathcal{A}_e(G) = \mathcal{A}_e^0(G)$ and $J_e(G) = J_e^0(G)$.

(a) Show that for all $B \in \mathcal{A}_e(G)$, we have that $\Omega_1(C_G(B)) = B$.

(b) Show that $J_e^i(G) \leq J_e^j(G)$ whenever $i \leqslant j$ and $i, j \in \{0, \ldots, m-1\}$.

(c) Show that $\Omega_1(C_G(J_e^i(G))) = \Omega_1(Z(J_e^i(G)))$ for all $i \in \{0, \ldots m-1\}$.

(d) Show that $\bigcap_{A \in \mathcal{A}_e(G)} A = \Omega_1(Z(J_e(G)))$.

(e) Show that if $J_e^i(G) \leq H \leq G$, then $J_e^i(G) = J_e^i(H)$ for all $i \in \{0, \ldots, m-1\}$.

Question 4.

Cook up a characteristic subgroup of your own accord. What properties does it satisfy? Is it always isomorphic to a known characteristic group?