Exercise Sheet 2

Question 1.

(a) Let G be a regular finite p-group. Show that every subgroup and every quotient of G is also regular.

(b) Prove that a finite 2-group is regular if and only if it is abelian.

(c) Let $G = \langle (1,2,3), (4,5,6), (7,8,9), (1,4,7)(2,5,8)(3,6,9) \rangle \leq \mathrm{Sym}(9)$.

  (i) Show that $G \cong C_3^3 \rtimes C_3$ (if you know the language, G is the wreath product of C_3 by C_3 written $C_3 \wr C_3$).

  (ii) Witness an element of G of order 9 and conclude that G is not regular.

(d) Let G be a finite regular p-group. Show that $[\Omega_i(G), \mho_j(G)] \leq \Omega_{i-j}(G)$ with the convention $\Omega_r(G) = \{e\}$ if $r \leqslant 0$.

(e) Show that either of the following conditions implies that a finite p-group G is regular:

  (i) $|G'/\mho_1(G')| < p^{p-1}$; or

  (ii) $|\Omega_1(G)| < p^{p-1}$.

Question 2.

(a) Prove that whenever $n > 3$, $\mathrm{SDih}(2^n)$ has exactly three maximal subgroups: a cyclic group $\langle x \rangle$, a dihedral group $\langle x^2, y \rangle$, and a *generalized quaternion group* $\langle x^2, xy \rangle$.

(b) Determine $Z(\mathrm{Mod}(p^n))$, $\Omega_1(\mathrm{Mod}(p^n))$ and $\mathrm{Mod}(p^n)'$ when $n > 3$. Deduce that $\mathrm{Mod}(p^n)$ has class 2 whenever $n > 3$.

(c) Describe the characteristic subgroups of Q_8, and compare them to the characteristic subgroups of Q_{2^n} when $n > 3$.

Question 3.

(a) Let G be a finite p-group. Say that G is *semi-extraspecial* if for every maximal subgroup Z of $Z(G)$, we have that G/Z is an extraspecial group. (We have that convention that extraspecial groups are also semi-extraspecial).

  (i) Show that if $K < G'$, then G/K is semi-extraspecial.

  (ii) Show that G is a special group.

  (iii) Show that if $|G/G'| = p^{2a}$ and $|G'| = p^b$, then $b \leqslant a$.

(b) Describe the non-abelian groups of order p^3 for all primes p.

(c) Show that $\mathrm{Dih}(8) \circ C_{2^n} \cong Q_8 \circ C_{2^n}$.

(d) Show that $Q_8 \circ Q_8 \cong \mathrm{Dih}(8) \circ \mathrm{Dih}(8) \not\cong Q_8 \circ \mathrm{Dih}(8)$ and $p_+^{1+2} \circ p_-^{1+2} \cong p_-^{1+2} \circ p_-^{1+2} \not\cong p_+^{1+2} \circ p_+^{1+2}$.