Exercise Sheet 1

Throughout, let G be a non-trivial finite p-group.

Question 1.

(a) Assume that $\{e\} \neq N \trianglelefteq G$. Show that $N \cap Z(G) \neq \{e\}$.

(b) Assume that $|G| = p^n$ and $N \trianglelefteq G$ of order p^a for some $a, n \in \mathbb{N}$ with $a \leqslant n$. Show that there is a chain of subgroups $\{e\} = G_0 < G_1 < \cdots < G_n = G$ such that each $G_i \trianglelefteq G$, $N = G_a$ and $|G_i/G_{i-1}| = p$ for $1 \leqslant i \leqslant n$.

Question 2.

(a) Suppose there is $x, y \in G$ with $G = \langle x, y \rangle$. Show that $G' = \langle [x, y]^g \mid g \in G \rangle$.

(b) Let $M, N \trianglelefteq G$ be such that $M = N[M, G]$. Show that $M = N$.

(c) Show that $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$.

(d) Give an example of a p-group whose upper and lower central series have distinct terms.

Question 3.

(a) Show that a 2-group of exponent 2 is abelian.

(b) Assume that $M < G$ is the unique maximal subgroup of G. What can you deduce about G?

(c) Let $A \leq G$ be abelian of index p.

  (i) Let $x \in G$. Show that the map $\phi_x : A \to A$ where $\phi_x(a) = [a, x]$ for $a \in A$ is a homomorphism.

  (ii) Conclude that $|G/Z(G)| = p|G'|$.

  (iii) Assume that A is not the unique abelian subgroup of index p in G. How many abelian subgroups of index p does G have?

(d) Assume that G has an abelian subgroup of index p^2. Prove that G has a normal abelian subgroup of index p^2.